1 Speed Mathematics

Who was Trachtenberg?

- Professor Jakow Trachtenberg was the founder of the Mathematical Institute in Zurich, Switzerland.
- He was a Russian, born June 17th, 1888 and studied engineering.
- While still in his early twenties, he became Chief Engineer with 11,000 men under his supervision.
- After the Czar of Russia was overthrown, he escaped to Germany where he became very critical of Hitler. He was later imprisoned.
- Most fellow prisoners around him gave up hope and died even before being sent to their death. He realized that if he wanted to stay alive, he had to occupy his mind with something else rather than focus on the hopeless conditions surrounding them. He set his mind on developing methods to perform speed mathematics.
- With the help of his wife, he escaped from prison and fled to Switzerland.
- There, he taught his speed math system to young children. It was very successful.

Trachtenberg developed a set of rules (algorithms) to multiply long numbers by numbers from 0 to 12. These rules allow one to dispense with memorizing multiplication tables, if that is desired. Even better, it gives a way to help memorize them, by allowing one to work out the answer by rule if one cannot remember it by rote. We perform each rule starting at the far right. The 'number' is the digit of the multiplicand just above the place that we are currently computing. The 'neighbour' is the digit immediately to the right of the 'number'. When there is no neighbour, we assume it is zero. We also write a zero in front of the multiplicand.

Note that the following rules only use the operations of addition, subtraction, doubling, and 'halving'.

Multiplier	**Rules**
0	Zero times any number at all is zero.
1	Copy down the multiplicand unchanged.
2	Double each digit of the multiplicand.
3	First step: subtract from 10 and double, and add 5 if the number is odd. Middle steps: subtract from 9 and double, and add half the neighbour, plus 5 if the number is odd. Last step: take half the lefthand digit of the multiplicand and reduce by 2.
4	First step: subtract from 10, and add 5 if the number is odd. Middle steps: subtract from 9 and add half the neighbour, plus 5 if the number is odd. Last step: take half the lefthand digit of the multiplicand and reduce by 1.
5	Use half the neighbour, plus 5 if the number is odd.
6	Use the number plus half the neighbour, plus five if the number is odd.
7	Use double the number plus half the neighbour, plus five if the number is odd.
8	First step: subtract from 10 and double. Middle steps: subtract from 9, double, and add the neighbour. Last step: Reduce the lefthand digit of the multiplicand by 2.
9	First step: subtract from 10. Middle steps: subtract from 9 and add the neighbour. Last step: reduce the lefthand digit of the multiplicand by 1.
10	Use the neighbour.
11	Add the neighbour to the number.
12	Double the number and add the neighbour.

Consider the following example:

$$\begin{array}{c} \underline{0\ 3\ 4\ 2\ 1\ 6\ 3\ 4} \quad \times \quad 1\ 1 \\ \textit{answer here} \end{array}$$

We use the rule **Add the neighbour to the number**. Starting at the far right, and looking at the 'number' 4, there is no neighbour to the right so we treat it as being a 0. Hence, $4 + 0$ is 4.

$$\begin{array}{rcl} \underline{0\ 3\ 4\ 2\ 1\ 6\ 3\ 4} & \times & 1\ 1 \\ 4 & & \end{array}$$

Now we look at the next position and the 'number' is 3. Adding 3 to it's neighbour of 4 gives 7:

$$\begin{array}{rcl} \underline{0\ 3\ 4\ 2\ 1\ 6\ 3\ 4} & \times & 1\ 1 \\ 7\ 4 & & \end{array}$$

Repeating gives:

$$\begin{array}{rcl} \underline{0\ 3\ 4\ 2\ 1\ 6\ 3\ 4} & \times & 1\ 1 \\ 3\ 7\ 6\ 3\ 7\ 9\ 7\ 4 & & \end{array}$$

Notice that we were able to write down the answer immediately, without having to write down any intermediate steps. The methods presented by Trachtenberg allow us to do this.

One thing to mention is carrying ones:

Consider the following example:

$$\begin{array}{rcl} \underline{0\ 3\ 7\ 7} & \times & 1\ 1 \\ \textit{answer} & & \end{array}$$

After the first step we get:

$$\begin{array}{rcl} \underline{0\ 3\ 7\ 7} & \times & 1\ 1 \\ 7 & & \end{array}$$

Now looking at the number 7 and adding it to its neighbour of 7 gives 14. Thus, we must carry this 1 to the next position.

$$\begin{array}{rcl} \underline{0\ 3\ 7\ 7} & \times & 1\ 1 \\ {}^{\cdot}4\ 7 & & \end{array}$$

We use a dot to indicate that there is a carry. Now the next number is 3 and adding it to its neighbour of 7 gives 10. Plus the carry gives 11.

$$\begin{array}{rcl} \underline{0\ 3\ 7\ 7} & \times & 1\ 1 \\ {}^{\cdot}1\ 4\ 7 & & \end{array}$$

Finally, the number 0 plus its neighbour of 3 gives 3, adding the carry gives 4.

$$\begin{array}{rcl} \underline{0\ 3\ 7\ 7} & \times & 1\ 1 \\ 4\ 1\ 4\ 7 & & \end{array}$$

What about multiplying by 12? The rule is **Double the number and add the neighbour**.

Consider the following example:

$$\begin{array}{rcl} \underline{0\ 3\ 4\ 2\ 1\ 6\ 3\ 4} & \times & 1\ 2 \\ \textit{answer here} & & \end{array}$$

Starting at the far right, we double the number 4 and add it to its neighbour (assumed to be 0):

$$\begin{array}{rcl} \underline{0\ 3\ 4\ 2\ 1\ 6\ 3\ 4} & \times & 1\ 2 \\ 8 & & \end{array}$$

We next double the number 3 and add it to its neighbour of 4 to get 10:

$$\begin{array}{rcl} \underline{0\ 3\ 4\ 2\ 1\ 6\ 3\ 4} & \times & 1\ 2 \\ {}^{\cdot}0\ 8 & & \end{array}$$

Now double the 6 and add it to 3, plus the carry gives:

$$\begin{array}{rcl} \underline{0\ 3\ 4\ 2\ 1\ 6\ 3\ 4} & \times & 1\ 2 \\ {}^{\cdot}6\ 0\ 8 & & \end{array}$$

Continuing gives:

$$\begin{array}{rcl} \underline{0\ 3\ 4\ 2\ 1\ 6\ 3\ 4} & \times & 1\ 2 \\ 4\ 1\ 0\ 5\ 9\ 6\ 0\ 8 & & \end{array}$$

Multiplication by 5, 6 or 7 make use of 'halving'. This isn't half of the number in the usual sense because if there is a fractional part left we throw it away. Mathematically speaking, 'halving' in the Trachtenberg system means $\lfloor x/2 \rfloor$. With a bit of practice, this process is to be done *instantly*.

Consider the following example:

$$\begin{array}{rcl} \underline{0\ 3\ 4\ 2\ 1\ 6\ 3\ 4} & \times & 6 \\ \textit{answer} & & \end{array}$$

The rule for 6 is **Use the number plus half the neighbour, plus five if the number is odd.**. Let us start with the number 4, add half of its neighbour (which is 0). This gives 4:

$$\begin{array}{rcl} \underline{0\ 3\ 4\ 2\ 1\ 6\ 3\ 4} & \times & 6 \\ 4 & & \end{array}$$

The next number is 3, and adding half of its neighbour 4 gives $3 + 2$ which is 5. But the rule for 6 says **plus five if the number is odd**. Our 'number' is 3 for this position, so we must add 5 to get 10:

$$\begin{array}{rcl} \underline{0\ 3\ 4\ 2\ 1\ 6\ 3\ 4} & \times & 6 \\ {}^{\cdot}0\ 4 & & \end{array}$$

Next we use the 'number' 6 and add half its neighbour, so add 1, then add the carry to get 8:

$$\begin{array}{rcl} \underline{0\ 3\ 4\ 2\ 1\ 6\ 3\ 4} & \times & 6 \\ 8\ 0\ 4 & & \end{array}$$

Repeating gives:

$$\begin{array}{rcl} \underline{0\ 3\ 4\ 2\ 1\ 6\ 3\ 4} & \times & 6 \\ 2\ 0\ 5\ 2\ 9\ 8\ 0\ 4 & & \end{array}$$

Rapid multiplication by the direct method

Trachtenberg developed two methods to multiply a large number by another large number. He calls them the 'direct method' and the 'two-finger method'. We will first go through the direct method.

The method for general multiplication is a method to achieve multiplication of $x \cdot y$ with as few temporary results as possible to be kept in memory. This is achieved by noting that the final digit is completely determined by multiplying the last digit of the multiplicands. This is held as a temporary result. To find the next to last digit, we need everything that influences this digit: The temporary result, the last digit of x times the next-to-last digit of y, as well as the next-to-last digit of x times the last digit of y. This calculation is performed, and we have a temporary result that is correct in the final two digits.

In general, for each position n in the final result, we compute

$$\sum_i x_i y_{n-i+1},$$

where x_i and y_j represents the ith digit of the number x and jth digit of the number y respectively. In other words, we write x as $x_k \cdots x_2 x_1$ and y as $y_m \cdots y_2 y_1$. If there is a carry from the $n-1$ position, we add the above sum to the carry and the last digit of this answer will be the nth position of xy (while the digits to the left of the last one are a carry). We treat both x and y as numbers with 0's in front of them, so that x_i and y_i are defined for each i from 1 to n, where n is the number of digits in the answer.

Trachtenberg didn't write down any formulas for his method, but instead drew pictures to help learn it. Mathematically, his method would look something like this: The nth digit of the product $x_k \cdots x_2 x_1$ times $y_m \cdots y_2 y_1$ is:

$$d_n \equiv \left[\sum_i x_i y_{n-i+1} + \text{carry}(d_{n-1}) \right] \mod 10,$$

where $x_i = 0$ if $i > k$ and $y_j = 0$ if $j > m$.

People can learn this algorithm and multiply 4 digit numbers in their heads, writing down only the final result, with the last digit first.

Consider the example:

$$\begin{array}{c} \underline{0\ 4\ 1\ 2} \\ \textit{answer} \end{array} \quad \times \quad 1\ 3$$

Here we have

$$d_1 = x_1 y_1 = 2 \cdot 3 = 6$$

Also,

$$d_2 = x_1 y_2 + x_2 y_1 = 2 \cdot 1 + 1 \cdot 3 = 5$$

For d_3 we have:

$$x_1 y_3 + x_2 y_2 + x_3 y_1 = 0 + 1 + 12 = 13,$$

thus $d_3 = 3$ and we must carry a 1. Finally,

$$d_4 = 0 + 4 + 1 = 5.$$

Thus, the answer is: 5356. Trachtenberg did not actually write any of these computation down, instead he mentally pictured arrows of which digits to multiply and add.

$$\begin{array}{cccc} & & & * \\ \underline{0} & \underline{4} & \underline{1} & \underline{2} \\ & & & 6 \end{array} \quad \times \quad \begin{array}{cc} & * \\ 1 & 3 \\ & \end{array}$$

To begin he drew a line from 2 to 3 and multiplied these numbers together to get 6: Next he considers the number 1 which has a neighbour of 2 and looks at the outside pairs. He draws an arrow from the number to the very end of the multiplier, then draws an arrow from the neighbour to the next to last digit of the multiplier. Multiplying using the arrows and adding gives 5:

$$\begin{array}{cccc} & & + & * \\ \underline{0} & \underline{4} & \underline{1} & \underline{2} \\ & & 5 & 6 \end{array} \quad \times \quad \begin{array}{cc} * & + \\ 1 & 3 \\ & \end{array}$$

Now consider the number 4 and move the arrows above the multiplicand one position to the left. The sum of the products gives 13:

$$\begin{array}{cccc} & + & * & \\ \underline{0} & \underline{4} & \underline{1} & \underline{2} \\ & \dot{3} & 5 & 6 \end{array} \quad \times \quad \begin{array}{cc} * & + \\ 1 & 3 \\ & \end{array}$$

Finally, we consider the number 0 and its neighbour 4. Then 0 times 3 is 0, and adding 4 times 1 gives 4, plus the remainder gives 5.

$$\begin{array}{cccc} + & * & & \\ \underline{0} & \underline{4} & \underline{1} & \underline{2} \\ 5 & 3 & 5 & 6 \end{array} \quad \times \quad \begin{array}{cc} * & + \\ 1 & 3 \\ & \end{array}$$

It is clear that this diagram can generalize for the situation:

$$\begin{array}{rcl} \underline{0\,0\,4\,1\,2\,3\,2\,1\,1\,3} & \times & 2\,1\,1 \\ \textit{answer here} & & \end{array}$$

Suppose we had the following and we wanted the 5th digit of the answer.

$$\begin{array}{rcl} \underline{0\,0\,4\,1\,2\,3\,2\,1\,1\,3} & \times & 2\,1\,1 \\ 5\,8\,4\,3 & & \end{array}$$

Then adding our arrows gives:

$$\begin{array}{rcl} * + \$ \quad\quad & & \$ + * \\ \underline{0\,0\,4\,1\,2\,3\,2\,1\,1\,3} & \times & 2\,1\,1 \\ 5\,8\,4\,3 & & \end{array}$$

And $3 \cdot 1 + 2 \cdot 1 + 1 \cdot 2 = 7$. Note that there was no carry from the previous digit (or else we would have put a dot). Continuing gives:

$$\begin{array}{rcl} \underline{0\,0\,4\,1\,2\,3\,2\,1\,1\,3} & \times & 2\,1\,1 \\ 8\,6\,9\,9\,9\,7\,5\,8\,4\,3 & & \end{array}$$

The above technique allows us to write down the answer, from right to left, without having to include any intermediate work. The main drawback of this method occurs when the numbers are mainly made up of 7's, 8's and 9's, since we are likely to have large numbers to have to add up mentally and large numbers to carry.

To get around this, Trachtenberg came up with an improvement which he calls the 'two-finger' method. Trachtenberg defined this algorithm with a kind of pairwise multiplication where 2 digits are multiplied by 1 digit, essentially only keeping the middle digit of the result. By performing the above algorithm with this pairwise multiplication, even fewer temporary results need to be held.

We won't go through this method since it will take quite a bit more time to explain in full detail. Trachtenberg also devised methods for division, squares, and square roots.